On Maps which Preserve Almost Periodicity

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If G is a topological group, let us denote by $AP(G)$ the set of all almost periodic functions on G, i.e.

$AP(G) = \{f: G \to \mathbf{C}$; the set of translates ${}_y f$, where $y \in G$, is relatively compact in $(\mathcal{CB}(G), \|\cdot\|_\infty)\}$.

Definition. Let G_2 and G_1 be two topological groups. A map $\rho: G_2 \to G_1$ is said to be *almost periodicity preserving* (a.p.p.) if, for every $f \in AP(G_1)$, one has $f \circ \rho \in AP(G_2)$.

In this lecture we shall see what exactly are the a.p.p. maps, under some particular hypothesis on the groups, and we shall give sketches of proofs in the two most classical cases: I) $G_2 = G_1 = \mathbf{R}$; II) $G_2 = G_1 = \mathbf{Z}$. These two examples are typical for the more general situation of connected groups, and discrete groups respectively. It turns out that the results are quite different in these two cases.

I. *Case* $G_2 = G_1 = \mathbf{R}$

Examples of maps $\rho: \mathbf{R} \to \mathbf{R}$ which are a.p.p. are:

1°) the group homomorphisms $\rho: x \to ax$, where a is a real constant;

2°) $\rho = h$, where h is a real-valued almost periodic function on $\mathbf{R}$.

Conversely one has the following:

THEOREM 1: If $\rho: \mathbf{R} \to \mathbf{R}$ is a.p.p., then there exists $a \in \mathbf{R}$ and $h \in AP(\mathbf{R})$, such that $\rho(x) = ax + h(x)$.

Generalization. Suppose G_2 is an abelian locally compact *connected* group, and $G_1 = \mathbf{R}$; the same statement remains true, just replacing ax by $\sigma(x)$, where σ is any continuous group homomorphism of G_2 into $\mathbf{R}$. (Cf. [3]).

II. *Case* $G_2 = G_1 = \mathbf{Z}$

Let there be given an integer $p \geqq 1$, and for every $i = 0, 1, \ldots, p-1$, two integers a_i and b_i. For every $x \in \mathbf{Z}$, let us divide x by p, obtaining $x = pq + i$, where q is the quotient and i the rest, and put

$$\rho(x) = a_i q + b_i.$$

Definition. Such a $\rho: \mathbf{Z} \to \mathbf{Z}$ is called *piecewise affine* (of modulus p).

THEOREM 2. ρ: $\mathbf{Z} \to \mathbf{Z}$ is a.p.p. if and only if ρ is piecewise affine. (This result is implicitly in [2]).

III. *Generalizations to discrete groups* (Cf. [1])

Let G_2 and G_1 be two discrete groups, where G_1 is *abelian,* but not necessarily G_2. Let us denote by $\mathcal{F}(G_2)$ the set of all subgroups of G_2 which are invariant and *of finite index.*

Definition. ρ: $G_2 \to G_1$ is said to be *piecewise affine* if there exists a subgroup $H \in \mathcal{F}(G_2)$, some representatives $x_0, x_1, \ldots, x_{p-1}$ of the classes of G_2 modulo H, and for every $i = 0,1,\ldots,p-1$:

1°) a group homomorphism $\sigma_i = H \to G_1$;
2°) a fixed $b_i \in G_1$,

such that:

$$x = y\, x_i \quad , \quad y \in H \Rightarrow \rho(x) = \sigma_i(y) + b_i.$$

THEOREM 3. If G_2 is *finitely generated,* then ρ: $G_2 \to G_1$ is a.p.p. if and only if ρ is piecewise affine.

THEOREM 4. If G_2 is countable and *without proper invariant subgroups of finite index,* then ρ: $G_2 \to G_1$ is a.p.p. if and only if $\rho(x) = \sigma(x) + b$, where σ is a group homomorphism of G_2 into G_1, and $b \in G_1$.

Example of a G_2 such that $\mathcal{F}(G_2) = \{G_2\}$: the group of all permutations of $\mathbf{N}$ which act only on finitely many elements and are even on them; this group is countable and simple.

IV. *Sketch of proof of the Theorem 1* (Cf. [3])

Let ρ: $\mathbf{R} \to \mathbf{R}$ be a.p.p. The proof proceeds in 4 steps:

Step 1: ρ is uniformly continuous.

Step 2: there exists a constant C such that, for every $x \in \mathbf{R}$ and $y \in \mathbf{R}$, $|\rho(x+y) - \rho(x) - \rho(y)| \leqq C$.

Step 3: there exists a continuous group homomorphism σ: $\mathbf{R} \to \mathbf{R}$ (i.e. a dilation $x \to ax$) and a map ρ_1: $\mathbf{R} \to \mathbf{R}$ which is a.p.p. and *bounded* such that

$$\rho = \sigma + \rho_1.$$

Step 4: if ρ: $\mathbf{R} \to \mathbf{R}$ is a.p.p. and bounded, then $\rho \in AP(\mathbf{R})$.

The step 1 is quite natural. The step 4 is easy to prove, for instance passing through the Bohr compactification of **R.** We shall give proofs for steps 2 and 3.

Proof of Step 2: One can suppose $\rho(0) = 0$. Put $f(x) = e^{i\rho(x)}$ and $E = \{\tau \in \mathbf{R}; \|f_\tau - f\|_\infty \leqq 1\}$. Since $f \in AP(\mathbf{R})$, there exists a compact interval K centered at 0, such that $\mathbf{R} = E + K$. For every $\tau \in E$, $x \in \mathbf{R}$,

$$|e^{i[\rho(x+\tau) - \rho(x)]} - 1| \leqq 1,$$

hence

$$\rho(x+\tau) - \rho(x) = m(\tau, x) + 2\pi\, k(\tau, x) \tag{1}$$

where $|m(\tau, x)| \leqq \frac{\pi}{3}$, and $k(\tau, x) \in \mathbf{Z}$. But for fixed τ, the first member of (1) is continuous in x on $\mathbf{R}$ (connected); hence $k(\tau, x) = k(\tau)$ does not depend on x. Notice that

$$C_1 = \sup\{|\rho(x) - \rho(y)|;\ x-y \in K\} \text{ is } < +\infty,$$

because ρ is uniformly continuous and K is compact.

Now for any $x \in \mathbf{R}$, $y \in \mathbf{R}$, choose $\tau \in E$ such that $\tau - x \in K$. We have

$$\begin{aligned} &|\rho(x+y) - \rho(x) - \rho(y)| \\ &\leqq |\rho(x+y) - \rho(\tau+y)| + |\rho(\tau+y) - \rho(y) - [\rho(\tau) - \rho(0)]| + |\rho(\tau) - \rho(x)| \\ &\leqq C_1 + |m(\tau, y) + 2\pi\, k(\tau) - m(\tau, 0) - 2\pi\, k(\tau)| + C_1 \\ &\leqq 2C_1 + \frac{2\pi}{3} = C. \end{aligned}$$

Proof of Step 3. Let M be a (Banach) invariant mean on $\mathcal{CB}(\mathbf{R})$. According to Step 2, for every fixed $y \in \mathbf{R}$, the function $x \to \rho(x+y) - \rho(x)$ is bounded continuous on $\mathbf{R}$. We put:

$$\sigma(y) = M_x(\rho(x+y) - \rho(x)).$$

From the invariance property of the mean M results that $\sigma(y+z) = \sigma(y) + \sigma(z)$. Hence σ: $\mathbf{R} \to \mathbf{R}$ is a homomorphism, continuous because ρ is uniformly continuous. Evidently $\rho_1 = \rho - \sigma$ is a.p.p., and ρ_1 is bounded, since:

$$\begin{aligned} |\rho_1(y)| &= |\rho(y) - \sigma(y)| = |\rho(y) - M_x(\rho(x+y) - \rho(x))| = \\ &= M_x(\rho(y) + \rho(x) - \rho(x+y)) \leqq C\, M_x(1) = C. \end{aligned}$$

V. *Sketch of proof of Theorem 2*

Let $\mathbf{T}$ be the set of $z \in \mathbf{C}$, $|z| = 1$. For every $z \in \mathbf{T}$, the function $x \to z^{\rho(x)}$ is almost periodic on $\mathbf{Z}$; in particular: $\forall z \in \mathbf{T}$, $\exists$ an integer $k_z > 0$ such that $\sup_{x \in \mathbf{Z}} |z^{\rho(x)} - z^{\rho(x+k_z)}| \leqq 1$.

Applying the Baire theorem we see that:

$$\left.\begin{array}{l}\exists \text{ an integer } k_z > 0, \exists \text{ a set } S \subset \mathbf{T}, \text{ where } S \text{ is open and not empty,} \\ \text{such that } \forall z \in S, \forall x \in \mathbf{Z}, |1 - x^{\rho(x+k)-\rho(x)}| \leqq 1.\end{array}\right\} \quad (2)$$

Necessarily the sequence $x \to \rho(x+k) - \rho(x)$ has only finite many values, because, if not, after H. Weyl, for a dense set of values of z, the sequence $x \to z^{\rho(x+k)-\rho(x)}$ should be dense in $\mathbf{T}$; but this is not true for $z \in S$ because of (2).

Choose $z \neq \sqrt[n]{1}$. The almost periodic function

$$x \to z^{\rho(x+k)-\rho(x)}$$

takes only finitely many values; hence there exists $q \in \mathbf{Z}$, $q > 0$, such that $x \to \rho(x+k) - \rho(x)$ is constant on every class of $\mathbf{Z}$ modulo q. Working a little more, we can conclude that ρ is piecewise affine.

VI. *The crucial lemma for the proofs of Theorems 3 and 4.* (Cf. [1])

In the proof for $G_2 = G_1 = \mathbf{Z}$ we had the great simplification that every subgroup $\neq \{e\}$ of $G_2 = \mathbf{Z}$ is automatically of finite index. Under the more general conditions of Theorems 3 and 4, we need to prove directly that some subgroup of G_2, which occurs in the proof, is *in fact* of finite index. For that purpose I proved the following lemma, which perhaps has its own interest:

Finiteness lemma: Let $s = \{n_1 < n_2 < \ldots < n_k < \ldots\}$ be an increasing sequence of positive integers. Suppose that

$$\limsup_{n \to \infty} \frac{\upsilon(n)}{n} > 0,$$

where $\nu(n)$ is the number of integers $\leqq n$ in the sequence s.

If $\varepsilon > 0$, let $Z(\varepsilon) = \{z \in \mathbf{T};$ for every $n_k \in s$, $|1 - z^{n_k}| \leqq \varepsilon\}$.

Then there exists $\varepsilon_0 > 0$, not depending on s, such that $Z(\varepsilon)$ is *finite* for $\varepsilon \leqq \varepsilon_0$.

VII. *Interpretation in terms of Bohr compactification*

If G is a topological group, there exists a compact group $\overline{G}$ and a continuous group homomorphism $\beta: G \to \overline{G}$, such that $\beta(G)$ is dense in $\overline{G}$, and such that: $f \in AP(G) \Leftrightarrow$ there exists $\bar{f} \in \mathscr{C}(\overline{G})$ with $f = \bar{f} \circ \beta$.

Let G_2 and G_1 be two locally compact groups, with G_1 abelian. Then a map $\rho: G_2 \to G_1$ is a.p.p. if and only if there exists a continuous map $\bar{\rho}: \overline{G}_2 \to \overline{G}_1$ such that the diagram

$$\begin{array}{ccc} \overline{G}_2 & \xrightarrow{\bar{\rho}} & \overline{G}_1 \\ \beta_2 \uparrow & & \uparrow \beta_1 \\ G_2 & \xrightarrow{\rho} & G_1 \end{array}$$

is commutative.

This interpretation gives curious consequences of the theorems above, concerning the *analysis situs* of groups in their Bohr compactification. For instance:

Corollary 1. Let $\overline{\mathbf{Z}} = [(\mathbf{T})_d]^\wedge$ be the dual group of the discrete torus. If a continuous map of $\overline{\mathbf{Z}}$ into $\overline{\mathbf{Z}}$ carries $\mathbf{Z}$ into $\mathbf{Z}$, then the restriction of this map to $\mathbf{Z}$ is piecewise affine.

Corollary 2. Let G_2 and G_1 be discrete groups, where G_1 is abelian, and G_2 countable with $\overline{G}_2$ connected. If a map $\rho: G_2 \to G_1$ such that $\rho(e) = 0$ can be extended to a continuous map from $\overline{G}_2$ into $\overline{G}_1$, then necessarily ρ is a group homomorphism.

VIII. *A problem*

The hypothesis G_1 abelian in Theorems 3 and 4 is not very aesthetic. To determine the a.p.p. maps of the free group with two generators into itself seems to me an interesting problem to attack now.

References

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